SHORT METHOD OF ELLIPTIC FUNCTIONS.

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(*Continued from page* 168, *Vol. IV.*)

NOTE 1. The inverse formula demonstrated in this Section, $\varDelta = \sqrt{b} \times \theta_3 x \div \theta x$ virtually accomplishes the object of various elaborate Researches, for the addition, subtraction, multiplication or division of Elliptic Functions; or for finding the roots of modular equations. For example, let it be required to resolve the equation, $m.F(e, \theta) = F(e, \theta') + F(e, \theta'') + F(e, \theta''')$; or $mAx = A(x' + x'' + x''')$; or $mx = x' + x'' + x'''$. The value of x, found through x', x'', x''', and substituted in the inverse formula will evidently define the unknown amplitude θ, of the function $mF(e, \theta)$ equal to the sum of the other three. Since $A''y = Ax'$; where $x' = A''y \div A$; different moduli can be included in the operation also.

NOTE 2. Double Periodicity: $\sin x = \sin(x + 2m\pi)$; $\sin x = \sin(x + n \log q.\sqrt{-1})$ or $\sin x = \sin(x + 2m\pi - n \log q.\sqrt{-1})$; where m, n are arbitrary integers; 2π is the *real* period, as in trigonometry, and $-\log q.\sqrt{-1}$, the *imaginary* period.

NOTE 3. The meaning of *Theta function* is thus defined:

$$\theta x = \Theta(x) = 1 - 2q \cos 2x + 2q^4 \cos 4x - 2q^9 \cos 6x + \ldots.$$

$$\theta_3 x = \Theta(\tfrac{\pi}{2} - x) = 1 + 2q \cos 2x + 2q^4 \cos 4x + 2q^9 \cos 6x + \ldots.$$

$$\Theta(0) = \sqrt{(bA)}; \qquad \Theta(\tfrac{\pi}{2}) = \sqrt{A}.$$

The last two equations are readily verified by taking the square root of the series for A near the close of Section V, and changing from terms of t to terms of q. Among other formulas from Abel and Jacobi, we find,

$$\theta_1 x = H(x) = 2q^{\frac{1}{4}} \sin x - 2q^{\frac{9}{4}} \sin 3x + 2q^{\frac{25}{4}} \sin 5x - \ldots.$$

$$\theta_2 x = H(\tfrac{\pi}{2} - x) = 2q^{\frac{1}{4}} \cos x + 2q^{\frac{9}{4}} \cos 3x + 2q^{\frac{25}{4}} \cos 5x + \ldots.$$

$$\sin\theta = \frac{1}{\sqrt{e}} \cdot \frac{\theta_1 x}{\theta x} = \frac{4}{eA}\left[\frac{q^{\frac{1}{2}} \sin x}{1-q} + \frac{q^{\frac{3}{2}} \sin 3x}{1-q^3} + \frac{q^{\frac{5}{2}} \sin 5x}{1-q^5} + \ldots\right].$$

$$\cos\theta = \sqrt{\frac{b}{e}} \cdot \frac{\theta_2 x}{\theta x} = \frac{4}{eA}\left[\frac{q^{\frac{1}{2}} \cos x}{1+q} + \frac{q^{\frac{3}{2}} \cos 3x}{1+q^3} + \frac{q^{\frac{5}{2}} \cos 5x}{1+q^5} + \ldots\right].$$

$$\varDelta = \sqrt{b} \cdot \frac{\theta_3 x}{\theta x} = \frac{1}{A} + \frac{4}{A}\left[\frac{q \cos 2x}{1+q^2} + \frac{q^2 \cos 4x}{1+q^4} + \ldots \frac{q^n \cos 2nx}{1+q^{2n}} + \ldots\right].$$

$$\theta = \int \varDelta\, dx = x + 2\left[\frac{q \sin 2x}{1+q^2} + \frac{1}{2}\frac{q^2 \sin 4x}{1+q^4} + \ldots \frac{1}{n}\frac{q^n \sin 2nx}{1+q^{2n}} + \ldots\right].$$

$$q = \frac{e^2}{16}\left(1 + \frac{1}{2}e^2 + \frac{21}{64}e^4 + \ldots\right). \quad \log q = -\frac{\pi A'}{A} = -\frac{\pi F(b, \frac{1}{2}\pi)}{F(e, \frac{1}{2}\pi)}.$$

$$\tfrac{1}{2}t = q - 2q^5 + 5q^9 - 10q^{13} + 18q^{17} - \ldots.$$

VII. SUCCESSIVE AND SIMULTANEOUS QUADRATURES.—A new scale of amplitudes, possessing important advantages, may be combined with the same scale of moduli already described in Sections III and IV. Thus let, in *descending* order,

$$\tan g = \sqrt[4]{(1-e^2)} = \sqrt{b};$$
$$\tan g^0 = \sqrt{(\sin 2g)} = \sqrt{b^0};$$
$$\tan g^{00} = \sqrt{(\sin 2g^0)} = \sqrt{b^{00}};$$
$$\tan g^{000} = \sqrt{(\sin 2g^{00})} = \sqrt{b^{000}};$$

$$\tan u = \sqrt{b}\,.\,\tan\theta, \qquad F = \int \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

$$\tan u^0 = \frac{\tan 2u}{\sqrt{b^0}}, \qquad F = \frac{2}{1+b}\int\frac{du}{\sqrt{(1-e^{02}\cos^2 2u)}}.$$

$$\tan u^{00} = \frac{\tan 2u^0}{\sqrt{b^{00}}}, \qquad F = \frac{2}{1+b}\cdot\frac{2}{1+b^0}\int\frac{\frac{1}{2}du^0}{\sqrt{(1-e^{002}\cos^2 2u^0)}}.$$

$$\tan u^{000} = \frac{\tan 2u^{00}}{\sqrt{b^{000}}}, \qquad F = \frac{2}{1+b}\cdot\frac{2}{1+b^0}\cdot\frac{2}{1+b^{00}}\int\frac{\frac{1}{4}du^{00}}{\sqrt{(1-e^{0002}\cos^2 2u^{00})}}.$$

. . . .

By way of demonstration we observe that,

$$d2u = \frac{d\tan 2u}{1+\tan^2 2u} = \frac{\sqrt{b^0}.d\tan u^0}{1+b^0\tan^2 u^0}. \quad \cos^2 2u = \frac{1}{1+\tan^2 2u} = \frac{1}{1+b^0\tan^2 u^0}.$$

Substituting in the second expression of F, since $b^{02} = 1-e^{02}$,

$$\frac{du}{\sqrt{(1-e^{02}\cos^2 2u)}} = \frac{\frac{1}{2}\sqrt{b^0}\,.\,d\tan u^0}{\sqrt{[(1+b^0\tan^2 u^0)(b^{02}+b^0\tan^2 u^0)]}}.$$

Now $\sqrt{b^0}$ may be divided out. Then since $d\tan u^0 = (1+\tan^2 u^0)du^0$, and $\tan^2 u^0 = \frac{1-\cos 2u^0}{1+\cos 2u^0}$, by trigonoetry,

$$\frac{du}{\sqrt{(1-e^{02}\cos^2 2u)}} = \frac{du^0}{\sqrt{\{[1+b^0+(1-b^0)\cos 2u^0].[1+b^0-(1-b^0)\cos 2u^0]\}}}.$$

The product of the sum and difference of the two quantities under the last radical, is known to be the difference of their squares. Passing out as a factor $(1+b^0)$, also multiplying by $2\times\frac{1}{2}$, and observing that $e^{00} = (1-b^0) \div (1+b^0)$, we obtain the third form of F above. By similar substitutions, the second expression of F was derived from the first; after which the law of derivation is evidently uniform.

On referring to the scale of amplitudes in Section IV, we observe that, if developed as in Section V, the leading periodic term of the integral there, may be denoted by $C\sin 2\theta^0$, or $C'\sin 2\theta^{00}$, ... In Section VII, the corresponding term is $-\frac{1}{2}C\sin 4u$, or $-\frac{1}{4}C'\sin 4u^0$,... So that the new scale

of amplitudes makes the periodic portion of the integral continually smaller than the former method, in the ratio of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, ...

And if, in the preceding denominators, we desire to change from a cosine to a sine, the relation of the first two equations will indicate the process. Thus in the fourth expression of F, let $\tan u^{00} = \sqrt{b^{00}} . \tan \Psi$; whence by comparison, $\tan \Psi = (\tan 2u^0) \div b^{00}$; and

$$F = \frac{2}{1+b} \cdot \frac{2}{1+b^0} \cdot \frac{1}{4} \int \frac{d\Psi}{\sqrt{(1-e^{002}\sin^2\Psi)}}.$$

In the preceding system, the amplitudes depend one upon another in *successive* order. Let us now represent the same amplitudes by *simultaneous* terms. For a *first* example, let T be such that

$$u^0 = \tan^{-1}(\tan T . \tan u) + \tan^{-1}(\cot T . \tan u).$$

Taking the trigonometric tangent of both sides

$$\tan u^0 = \frac{(\tan T + \cot T)\tan u}{1-\tan^2 u} = \frac{\tan 2u}{\sin 2T} = \frac{\tan 2u}{\sqrt{b^0}}.$$

Thus the auxiliary T is found by the relation $\sin 2T = \sqrt{b^0}$. And if

$$u^0 = u_1 + u_2,\ \tan u_1 = \tan T . \tan u,\ \tan u_2 = \cot T . \tan u.$$

For a *second* application, we assume

$$u^{00} = \tan^{-1}(\tan T^0 . \tan u^0) + \tan^{-1}(\cot T^0 . \tan u^0).$$

Taking the tangent and proceeding as before, we find $\sin 2T^0 = \sqrt{b^{00}}$. Also substituting for $\tan u^0$ its former equal $(\tan 2u) \div \sqrt{b^0}$,

$$u^{00} = \tan^{-1}\left(\frac{\tan 2u}{\tan T^0 . \sqrt{b^0}}\right) + \tan^{-1}\left(\frac{\tan 2u}{\cot T^0 . \sqrt{b^0}}\right).$$

Here let $\sin 2T_1 = \tan T^0 . \sqrt{b^0}$; $\sin 2T_2 = \cot T^0 . \sqrt{b^0}$. By comparison with the first example, u^{00} can now be resolved into the four simultaneous terms $u^{00} = u_1 + u_2 + u_3 + u_4$; where

$$\tan u_1 = \tan T_1 \tan u; \qquad \tan u_2 = \cot T_1 \tan u;$$
$$\tan u_3 = \tan T_2 \tan u; \qquad \tan u_4 = \cot T_2 \tan u.$$

In like manner for a *third* example, u^{000} can be resolved into eight similar terms, by the relations,

$$\sin 2T^{00} = \sqrt{b^{000}};\ \sin 2T_1^0 = \tan T^{00} . \sqrt{b^{00}};\ \sin 2T_2^0 = \cot T^{00} . \sqrt{b^{00}}.$$
$$\sin 2T_1 = \tan T_1^0 . \sqrt{b^0};\ \sin 2T_2 = \cot T_1^0 \sqrt{b^0};$$
$$\sin 2T_3 = \tan T_2^0 . \sqrt{b^0};\ \sin 2T_4 = \cot T_2^0 \sqrt{b^0}.$$
$$u^{000} = u_1 + \ldots + u_8;\ \tan u_1 = \tan T_1 \tan u;\ \tan u_2 = \cot T_1 \tan u; ..$$

For a *fourth* example, beginning with $\sin 2T^{000} = \sqrt{b^{0000}}$, we may extend the series to sixteen similar terms, whose sum is u^{0000}; and so on. In all the examples, $\tan u$ may be replaced by its equal $\sqrt{b} . \sin \theta$. And the computed value of u, or of u^0 or of u^{00}, etc., is to be substituted in the proper equation of F, of the series shown at the beginning of this Section.

VIII. Quadrantal Quadrature.—It will be instructive to observe how the preceding system and other known scales of moduli and amplitude naturally originate in the integration for F by quadrantal quadrature. Let us first represent the binomial coefficients by the well known integrals with respect to an auxiliary arc ω. Thus,

$$dF = \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = d\theta(1+\tfrac{1}{2}e^2\sin^2\theta+\tfrac{3}{8}e^4\sin^4\theta+\ldots),$$

$$dF = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\omega(1+\sin^2\omega.e^2\sin^2\theta+\sin^4\omega.e^4\sin^4\theta+\ldots)d\theta.$$

Taking the sum of this geometric series, and integrating in respect to θ,

$$dF = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\omega\, d\theta}{1-e^2\sin^2\omega\sin^2\theta},$$

$$F = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\omega}{\sqrt{(1-e^2\sin^2\omega)}}\tan^{-1}[\sqrt{(1-e^2\sin^2\omega)}.\tan\theta].$$

As heretofore, let F be replaced by Ax; let $\varDelta = \sqrt{(1-e^2\sin^2\omega)}$; then $\int d\omega \div \varDelta = A.\tfrac{1}{2}\pi = \int A dx'$. Dividing out A, and regarding ω or $\varDelta$ as an implicit function of x', we have

$$x = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} dx'.\tan^{-1}(\varDelta.\tan\theta).$$

While x denotes the true integral, let Ψ denote its approximate value as found by the common trapezoidal formulas of quadrature. Thus for the two sides of one trapezoid,

$$\Psi = \tfrac{1}{2}\tan^{-1}(\varDelta_0\tan\theta)+\tfrac{1}{2}\tan^{-1}(\varDelta_1\tan\theta), \text{ or}$$

$$\tan(2\Psi-\theta) = b\tan\theta;\ \varDelta_0 = 1,\ \varDelta_1 = b.$$

The original scale of Landen and Lagrange described in Sections III, IV, is here evidently reproduced in amplitude.

Taking the middle ordinate of one trapezoid,

$$\Psi = \tan^{-1}(\varDelta_{\frac{1}{2}}\tan\theta), \text{ or } \tan\Psi = \sqrt{b}\,.\tan\theta.$$

Here we recognize the first new amplitude u of Section VII. Again taking the middle ordinates in two trapezoids,

$$\Psi = \tfrac{1}{2}\tan^{-1}(\varDelta_{\frac{1}{4}}\tan\theta)+\tfrac{1}{2}\tan^{-1}(\varDelta_{\frac{3}{4}}\tan\theta).$$

Observing that $\varDelta_{\frac{1}{4}}$, $\varDelta_{\frac{3}{4}}$ are complementary, or $\varDelta_{\frac{1}{4}}\varDelta_{\frac{3}{4}} = b$; we have,

$$\tan 2\Psi = \frac{(\varDelta_{\frac{1}{4}}+\varDelta_{\frac{3}{4}})\tan\theta}{1-b\tan^2\theta} = \frac{(\varDelta_{\frac{1}{4}}+\varDelta_{\frac{3}{4}})}{2\sqrt{b}}\tan 2u = \frac{\tan 2u}{\sqrt{b^o}}.$$

This equation evidently coincides with that of the second new amplitude, and is the type of those that follow, in the beginning of Section VII.

Let us digress to verify the equality of the last two members. For if equal, $\varDelta_{\frac{1}{4}}+\varDelta_{\frac{3}{4}} = 2\sqrt{b}\div\sqrt{b^o} = \sqrt[4]{b}\,.\sqrt{[2(1+b)]}$; $\varDelta_{\frac{1}{4}}\varDelta_{\frac{3}{4}} = b$. Whence

$\varDelta_{\frac{1}{4}}$ is found by a quadratic, and $\varDelta^2_{\frac{1}{4}} = \sqrt{b}.[1+b-\sqrt{b}+(1-\sqrt{b})\sqrt{(1+b)}]$. The value of $\varDelta^2_{\frac{1}{4}}$ found by the independent method of bisection, being precisely the same, verifies the equality.

On taking the sides of two trapezoids, we again find the values of $\varDelta$ to occur in the geometric progression 1, $\sqrt{b}$, b; thus,

$$\varPsi = \theta + 2\tan^{-1}(\sqrt{b} \,.\, \tan\theta) + \tan^{-1}(b\tan\theta) = \tan^{-1}[(\tan 2u^o) \div b^{oo}].$$

Generally let p denote the number of trapezoids, p being an odd number. Taking the first side or θ with twice the other alternate sides, we have $\varPsi$ the amplitude of summation in the first Theorem of Jacobi. Thus,

$$\varPsi = \theta + 2\tan^{-1}(\varDelta_{\frac{2}{p}}\tan\theta) + \ldots . + 2\tan^{-1}(\varDelta_{\frac{p-1}{p}}\tan\theta);\ \text{or}$$

$$\varPsi = \theta + 2\theta_2 + 2\theta_4 + 2\theta_6 + \ldots . + 2\theta_{p-1}.$$

When $p = 3$, for examble, let the amplitudes a_1, a_2 be found for the relations $F(e, a_1) = \frac{1}{3}F(e, \frac{1}{2}\pi)$; $F(e, a_2) = \frac{2}{3}F(e, \frac{1}{2}\pi)$; which can be resolved by Legendre's Tables, or by Section VI. Generally let m denote any whole number less than p; then $F(e, a_m) = \frac{m}{p}F(e, \frac{1}{2}\pi)$.

$$F(e, \theta) = \beta F(h, \varPsi);\ \tan\theta_m = \varDelta_{\frac{m}{p}}\tan\theta = \frac{\cos a_m}{\sin a_{p-m}} \,.\, \tan\theta.$$

$$h = e^p \sin^4 a_1 \sin^4 a_3 \sin^4 a_5 \ldots . \sin^4 a_{p-2}\,.$$

$$\beta = \frac{\sin^2 a_1 \sin^2 a_3 \ldots . \sin^2 a_{p-2}}{\sin^2 a_2 \sin^2 a_4 \ldots . \sin^2 a_{p-1}} = \frac{p \,.\, F(h, \frac{1}{2}\pi)}{F(e, \frac{1}{2}\pi)}.$$

The decreased modulus h may also be found by trial from the equation

$$\frac{F(h, \frac{1}{2}\pi)}{F[\sqrt{(1-h^2)}, \frac{1}{2}\pi]} = \frac{F(e, \frac{1}{2}\pi)}{pF[\sqrt{(1-e^2)}, \frac{1}{2}\pi]}.$$

In the Philosophical Transactions for 1831, Mr. Ivory has simplified the demonstration of Jacobi's Theorem and extended it to the case where p is an even number, that is to regular trapezoidal quadrature. Thus when p is even,

$$\varPsi = \theta + 2\tan^{-1}(\varDelta_2\tan\theta) + \ldots . + 2\tan^{-1}(\varDelta_{p-2}\tan\theta) + \tan^{-1}(\varDelta_p\tan\theta).$$

The second Theorem of Jacobi can be applied to large moduli, by first changing e to b; and from $F(b, \theta)$ finding β', h', $\varPsi'$ according to a modification of the first Theorem; then changing back from h' to the larger h, where $h = \sqrt{(1-h'^2)}$, so that $F(e, \theta) = (1 \div \beta')F(h, \varPsi')$. When h becomes virtually 1, the integration is effected by logarithms.

In Section VI, was shown the discovery of Jacobi that the exponents of q in the Theta function increase as the series of squares 1, 4, 9, 16, . . . A comparison of results by Jacobi discloses another property equally remarkable. Let q, $''$, qq^{oo}, . . . denote the values of q respectively defined by b, b^o, b^{oo}, ... *in the decreasing scale of moduli.* Then rigorously $q = q$, $q^o = q^2$, $q^{oo} = q^4$, $q^{ooo} = q^8$, . . . That is, *the exponents of q increase regularly in geometrical progression.*

Thus in the descending scale of $\varphi = 45^\circ$, $9^\circ 52\frac{3}{4}'$, $25\frac{2}{3}'$, $2\frac{7}{8}''$,... Log q $= \bar{2}.63563\pm$; Log $q^o = \bar{3}.27127 =$ Log q^2; Log $q^{ooo} = \overline{11}.0850 =$ Log q^8.

The like property belongs to other scales of moduli; and it happily meets the difficulty of finding small arcs to a sufficient number of places from the cosine in the common logarithmic tables.

For large Moduli. Where e is near to 1, we may change to the complementary modulus by assuming as in Section II, $\cos\theta \,.\, \cos\theta' = 1$; thus if $\Delta' = \sqrt{(1-b^2\sin^2\omega)}$;

$$dF = \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = \frac{\sqrt{-1}\,.\,d\theta'}{\sqrt{(1-b^2\sin^2\theta')}} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sqrt{-1}.d\omega d\theta'}{1-b^2\sin^2\omega\sin^2\theta'},$$

$$F = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sqrt{-1}.d\omega}{\Delta'}\tan^{-1}(\Delta'\tan\theta') = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sqrt{-1}.d\omega}{\Delta'}\tan^{-1}\frac{\Delta'\sin\theta}{\sqrt{-1}}.$$

By applying common quadrature to the last member, dividing by the common coefficient of $\tan^{-1}$, and then taking the trigonometric tangent of both sides we might obtain formulas corresponding to the second Theorem of Jacobi. Or developing the are in terms of its tangent, we may apply quadrature to the following sum of the series:

$$F = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\omega}{\Delta'}\,.\,\tfrac{1}{2}\log\frac{1+\Delta'\sin\theta}{1-\Delta'\sin\theta}.$$

$$F = \sin\theta + \tfrac{1}{3}(1-\tfrac{1}{2}b^2)\sin^3\theta + \tfrac{1}{5}(1-b^2+\tfrac{3}{8}b^4)\sin^5\theta + \ldots.$$

$$= a\sin\theta + a_1\log\frac{1+a_2\sin\theta}{1-a_2\sin\theta}, \text{ nearly};$$

where a, a_1, a_2 are to be found from correlative values of F and θ, or from the first three coefficients of powers of $\sin\theta$. When more terms are required, the last formula but two will have the advantage of a common factor, by treating the coefficient of log as the variable of quadrature, and proceding with b, as in the former method with e.

IX. Integration of Functions of the Second and the Third Order. — The common methods of quadrature are occasionally employed for all the Elliptic Functions. For illustration, required the value of F when $e = \sin 45^\circ$, and $\theta = 90^\circ$. In a first trial, let the 90° be divided into three equal parts of 30° each. Computing for the middle of each part, by the formula for quadrature, we have

$$F(e, \tfrac{1}{2}\pi) = \frac{1}{3}\cdot\frac{\pi}{2}\cdot\left(\frac{1}{\sqrt{(1-e^2\sin^2 15^\circ)}} + \frac{1}{\sqrt{(1-e^2\sin^2 45^\circ)}} + \frac{1}{\sqrt{(1-e^2\sin^2 75^\circ)}}\right) + R.$$

The result so found is $1.854053 + R$. The true result is 1.854075; or the residue R is 0.000022. By dividing the amplitude into four or more parts, the integral may thus be found to any degree of accuracy.

Besides the two series for E already given in Section V, we here insert from Jacobi, another series in terms of A, q, x, as previously defined:

$$E(e,\theta)=\frac{2x\,.\,E(e,\frac{\pi}{2})}{\pi}+\frac{2\pi}{A}\cdot\frac{q\sin 2x-2q^4\sin 4x+3q^9\sin 6x-\ldots.}{1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+\ldots}$$

And in terms of the scale of Section IV,

$$E(e,\theta)=F(e,\theta)[1-\tfrac{1}{2}e^2(1+\tfrac{1}{2}e^{o}+\tfrac{1}{4}e^{o}e^{oo}+\tfrac{1}{8}e^{o}e^{oo}e^{ooo}+\tfrac{1}{16}e^{o}e^{oo}e^{ooo}e^{oooo}+\ldots)]$$
$$+e[\tfrac{1}{2}\sqrt{e^{o}}.\sin\varphi^{o}+\tfrac{1}{4}\sqrt{(e^{o}e^{oo})}.\sin\varphi^{oo}+\tfrac{1}{8}\sqrt{(e^{o}e^{oo}e^{ooo})}.\sin\varphi^{ooo}+\ldots].$$

Among formulas of quadrantal integration, the following is readily verified by differentiation with respect to θ:

$$E(e,\theta)=\cot\theta(1-\varDelta)+\frac{2}{\pi}\int_0^{\frac{\pi}{2}}d\omega\sqrt{(1-e^2\sin^2\omega)}.\tan^{-1}[\sqrt{(1-e^2\sin^2\omega)}.\tan\theta].$$

Lastly the value of E may be found by taking a sufficient number of simultaneous terms from Section VII or VIII to represent $dF=Ad\varPsi\div n$, and making the obvious substitution for $dF=d\theta\div\varDelta$,

$$dE=\sqrt{(1-e^2\sin^2\theta)}.d\theta\times\varDelta\div\varDelta=(1-e^2\sin^2\theta)A\times d\varPsi\div n.$$

Thus, when e is so small that two single, with one double term will suffice, we take

$$\varPsi=\theta+2\tan^{-1}(\sqrt{b}\,.\tan\theta)+\tan^{-1}(b\tan\theta).\quad\text{Then}$$
$$dE=(1-e^2\sin^2\theta)\tfrac{1}{4}A\left(1+\frac{2\sqrt{b}}{1-(1-b)\sin^2\theta}+\frac{b}{1-(1-b^2)\sin^2\theta}\right)d\theta;$$

the integral of which reduces to

$$E=\tfrac{1}{4}A[2(1+b)(1+b+4\sqrt{b})\theta+\tfrac{1}{4}e^2\sin 2\theta-2b\tan^{-1}(\sqrt{b}.\tan\theta)$$
$$+(1-b)\tan^{-1}(b\tan\theta)].$$

The integral of the Third Species $\varPi$ may be found in like manner. For illustration, when the above three terms will give a sufficient approximation; substituting for $dF=\frac{1}{4}A.d\varPsi$, we have, since $d\varPi=$

$$\frac{d\theta}{(1+n\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}};\ d\varPi=\frac{d\theta}{1+n\sin^2\theta}\left(1+\frac{2\sqrt{b}}{1-(1-b)\sin^2\theta}+\frac{b}{1-e^2\sin^2\theta}\right)\frac{A}{4}.$$

Integrating by the method of rational fractions,

$$\varPi=\frac{A}{4}\left[1+\frac{2n\sqrt{b}}{n+1-b}+\frac{nb}{n+e^2}\right]\frac{\tan^{-1}[\sqrt{(1+n)}.\tan\theta]}{\sqrt{(1+n)}}+\frac{A}{4}\left[\frac{2-2b}{1-b+n}\right.$$
$$\left.\times\tan^{-1}(\sqrt{b}\,.\tan\theta)+\frac{e^2\tan^{-1}(b\tan\theta)}{n+e^2}\right].$$

Among other formulas we next insert that of Jacobi: Let

$$+n=-e^2\sin^2a;\ F(e,\theta)=Ax;\ F(e,a)=Ax';\ \text{then,}$$

$$\varPi(n,e,\theta)=Ax+\frac{1}{\cot a.\varDelta(e,a)}\left[4x\left(\frac{q\sin 2x'}{1-q^2}+\frac{q^2\sin 4x'}{1-q^4}+\ldots\right)+\tfrac{1}{2}\log\frac{\Theta(x-x')}{\Theta(x+x')}\right]$$

Again from Legendre's *Traite\` des Fonctions Elliptiques*, vol. 1, p. 141;–

$$\varPi(n,e,\tfrac{1}{2}\pi)=F(e,\tfrac{1}{2}\pi)+[\tan a\div\sqrt{(1-e^2\sin^2a)}][F(e,\tfrac{1}{2}\pi)E(e,a)-E(e,\tfrac{1}{2}\pi)F(e,a)].$$

Here as before, $n=-e^2\sin^2a$.

Lastly in very convergent series, when either n or e is comparatively small, and the amplitude is $90°$; we first find

$$m=\frac{\sqrt{(1+n)}-1}{\sqrt{(1+n)}+1}\cdot\left(\frac{2}{e^2}-1\right);\ m'=\left(\frac{\sqrt{(1+n)}-1}{\sqrt{(1+n)}+1}\right)^2=k^2;$$

$A=\frac{2}{\pi}F(e,\frac{1}{2}\pi)$; $B=\frac{2}{\pi}E(e,\frac{1}{2}\pi)$; $p_1=2mA-(2m+2k)B$; $p_2=\frac{4}{3}mp_1$ $-\frac{2}{3}m'A$; $p_3=\frac{8}{5}mp_2-\frac{3}{5}m'p_1$; $p_4=\frac{12}{7}mp_3-\frac{5}{7}m'p_2$; ....

$$\Pi(n,e,\tfrac{\pi}{2})=\frac{\pi}{2\sqrt{(1+n)}}\Big[A-p_1+p_2-p_3+p_4-\ldots\Big]$$

For demonstration, we have from Analytic Trigonometry,

$$\frac{1}{1+n\sin^2\theta}=\frac{1}{\sqrt{(1+n)}}\left(1+2k\cos 2\theta+2k^2\cos 4\theta+2k^3\cos 6\theta+\ldots\right)$$

$$\frac{1}{\sqrt{(1-e^2\sin^2\theta)}}=A-2A_1\cos 2\theta+4A_2\cos 4\theta-6A_3\cos 6\theta+\ldots$$

Substituting, multiplying and integrating through a quadrant,

$$\Pi(n,e,\tfrac{\pi}{2})=\frac{\pi}{2\sqrt{(1+n)}}\left(A-2kA_1+4k^2A_2-6k^3A_3+\ldots\right)$$

Here k is put for $[\sqrt{(1+n)}-1]\div[\sqrt{(1+n)}+1]$; $p_1=2kA_1$; $p_2=4k^2A_2$; etc. And the above derivation of $p_1, p_2, \ldots$ is readily found from that of $A_1, A_2, \ldots$ shown in Section V.

The general integration, when the parameter n is greater than e, can be made to depend on a parameter less than e, by the following relation, where $nn'=e^2$; $(1+n)(1+n')=a$;

$$\Pi(n)+\Pi(n')=F+\frac{1}{\sqrt{a}}\tan^{-1}\frac{\sqrt{a}\,.\tan\theta}{\Delta}.$$

This can be verified by differentiation. When $n=-e^2$, and when $n=-1$, the integrals are shown at (16), (20), in Section I. When $n=n'=\pm e$,

$$\Pi(n)=\tfrac{1}{2}F+\frac{1}{2\sqrt{a}}\tan^{-1}\frac{\sqrt{a}\,.\tan\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

Again in the special case of $e=1$, we have

$$\Pi=\int_0^\theta\frac{d\theta}{(1-h\sin^2\theta)\cos\theta}=\frac{1}{2(1-h)}\left(\sqrt{h}.\log\frac{1-\sqrt{h}.\sin\theta}{1+\sqrt{h}.\sin\theta}+2\log\tan(45°+\tfrac{1}{2}\theta)\right).$$

Lastly by transformation, we have found the following integral, subject to the particular relation,

$$h=1\pm b=1\pm\sqrt{(1-e^2)};\ \Delta=\sqrt{(1-e^2\sin^2\theta)};$$

$$\Pi=\int_0^\theta\frac{d\theta}{(1-h\sin^2\theta)\Delta}=\frac{h-2}{2h-2}F(e,\theta)+\frac{1}{4h-4}\log\frac{\Delta+h\sin\theta\cos\theta}{\Delta-h\sin\theta\cos\theta}.$$

The logarithm evidently disappears when $\theta=90°$.